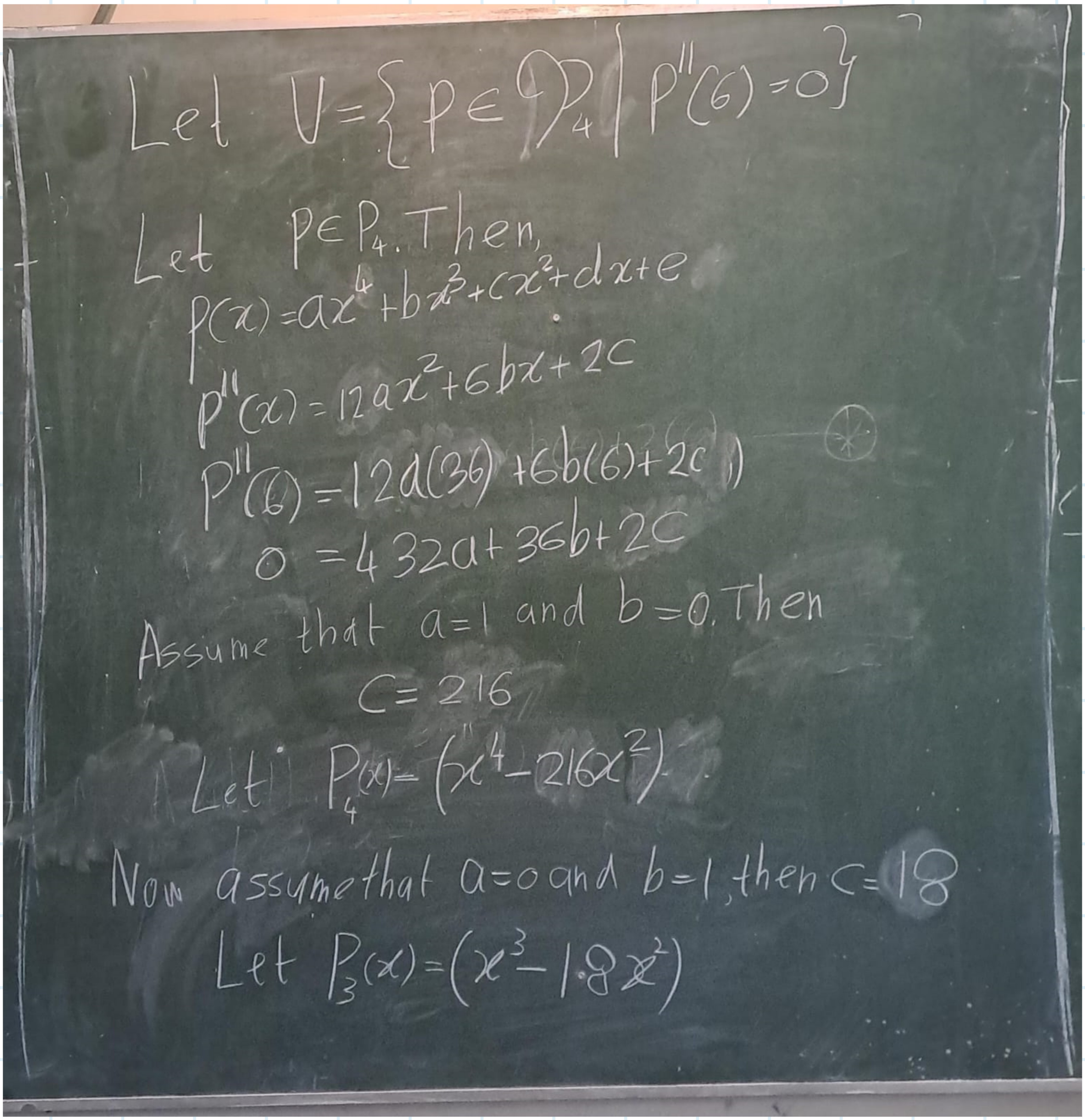
4 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.



Above roughwork, we found two possible candidates for basis. Furthur, we have to two more. So, let

$$P_1(x)=1 \Rightarrow P''(x)=0 \Rightarrow P_1''(6)=0 \text{ and}$$

$$P_2(x)=x \Rightarrow P_2''(x)=0 \Rightarrow P_2'(6)=0$$

All candidates are,

$$P_1(x)=1, \quad P_3(x)=(x^3-18x^2)$$

$$P_2(x)=x, \quad P_4(x)=(x^4-216x^2)$$

Claim: P_1, P_2, P_3, P_4 are linearly independent.

Suppose that $\exists\, a_1, \ldots, a_4 \in \mathbb{F}$ such that

$$a_1 + a_2 x + a_3(x^3 - 18x^2) + a_4(x^4 - 216x^2) = 0$$

By comparing relevent terms.

constant term $\longrightarrow$ $a_1 = 0$

x term $\longrightarrow$ $a_2 = 0$

x^3 term $\longrightarrow$ $a_3 = 0$

x^4 term $\longrightarrow$ $a_4 = 0$

Thus P_1, P_2, P_3, P_4 are linearly independant.

Claim 2: P_1, P_2, P_3, P_4 spans U.

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in U$.

$$= (ax^4 - 216ax^2) + 216ax^2 + (bx^3 - 18bx^2) + 18bx^2 + cx^2 + dx + e$$

$$= a(x^4 - 216x^2) + b(x^3 - 18x^2) + (216a + 18b)x^2 + cx^2 + dx + e$$

$$= aP_4(x) + bP_3(x) - cx^2 + cx^2 + dx + e$$

$$= aP_4(x) + bP_3(x) + dP_2(x) + eP_1(x)$$

Therfore $P_1, \dots, P_4$ is a basis for U.

b)

Now let $P_1(x) = 1$

$P_2(x) = x$

$P_3(x) = (x^3 - 18x^2)$

$P_4(x) = (x^4 - 216x^2)$

$P_5(x) = x^2$

Claim3: P_1, P_2, P_3, P_4, P_5 are linearly independant)

Suppose that $\exists\, a_1, \ldots, a_5 \in \mathbb{F}$ such that

$$a_1 + a_2 x + a_3(x^3 - 18x^2) + a_4(x^4 - 216x^2) + a_5 x^2 = 0$$

By comparing relevent terms.

constant term $\longrightarrow a_1 = 0$

x term $\longrightarrow a_2 = 0$

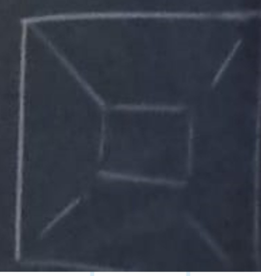
x^3 term $\longrightarrow a_3 = 0$

x^4 term $\longrightarrow a_4 = 0$

x^2 term $\longrightarrow -18a_3 - 216a_4 + a_5 = 0$

$\Longrightarrow a_5 = 0$

Thus P_1, P_2, P_3, P_4, P_5 are linearly independant.



Claim 4: $P_1, \dots, P_5$ spans $\mathcal{P}_4$

Let $g(x) = k_4x^4 + k_3x^3 + k_2x^2 + k_1x^1 + k_0$

$$= k_4x^4 - 216k_1x^2 + 216k_1x^2$$
$$+ k_3x^3 - 18k_3x^2 + 18k_3x^2$$
$$+ k_2x^2 + k_1x + k_0$$

$$= k_4(x^4 - 216x^2) + k_3(x^3 - 18x^2)$$
$$+ (216k_1 + 18k_3 + k_2)x^2$$
$$+ k_1x + k_0$$

Thus, $P_1, \dots, P_5$ span $\mathcal{P}_4$

Therefore, $P_1, \dots, P_5$ is a basis for $\mathcal{P}_4$

c)

Let $W = \text{span}\{x^2\} = \{kx^2 \mid k \in \mathbb{R}\}$

Then $U + W = \mathcal{P}_4$ (by b))

Furthur, It is easily see that $U \cap W = \{0\}$

Therefore, $U \oplus W = \mathcal{P}_4$